$$-\Phi_{11}(\mathcal{L})\Phi_{12}(\mathcal{L}) + \Phi_{12}(\mathcal{L})\Phi_{22}(\mathcal{L}) = {}_n0_n \tag{32c}$$

$$\Phi_{21}(\mathcal{L})\Phi_{11}(\mathcal{L}) - \Phi_{22}(\mathcal{L})\Phi_{21}(\mathcal{L}) = {}_n0_n. \tag{32d}$$

If $\Phi_{12}(\mathcal{L})$ and $\Phi_{21}(\mathcal{L})$ are nonsingular, then (32c) becomes Identity 3 and (32d) becomes Identity 4. Substituting these two identities into (32a) and (32b) we obtain Identity 1 and Identity 2, respectively. Note that Identities 1,2,3,4 do not rely on Z and Y being symmetric.

IV. INCORPORATING THE TERMINATION NETWORKS

The desired end result in an analysis of transmission line behavior is a determination of the voltages and currents along the line when the line is driven by linear termination networks at the ends of the line. For these purposes, we choose to characterize these linear termination networks by "generalized Thevenin equivalents" as

$$V(0) = E_0 - Z_0 I(0) \tag{33a}$$

$$V(\mathcal{L}) = E_{\mathcal{L}} + Z_{\mathcal{L}} I(\mathcal{L}) \tag{33b}$$

where E_0 and $E_{\mathcal{L}}$ are $n \times 1$ complex vectors of equivalent open circuit port excitation voltages (with respect to the reference conductor) and Z_0 and $Z_{\mathcal{L}}$ are $n \times n$ complex symmetric impedance matrices. The matrix chain parameters relate the voltages and currents at the ends of the line as

$$V(\mathcal{L}) = \Phi_{11}(\mathcal{L})V(0) + \Phi_{12}(\mathcal{L})I(0) \tag{34a}$$

$$I(\mathcal{L}) = \Phi_{21}(\mathcal{L})V(0) + \Phi_{22}(\mathcal{L})I(0). \tag{34b}$$

The objective now is to eliminate $V(0)$ and $V(\mathcal{L})$ from (33) and (34) to yield a set of $2n$ equations in the $2n$ unknowns $I(0)$ and $I(\mathcal{L})$. Substituting (33a) into (34b) and rearranging yields the first equation

$$I(\mathcal{L}) + (\Phi_{21}Z_0 - \Phi_{22})I(0) = \Phi_{21}E_0. \tag{35}$$

The remaining equation can be obtained by substituting (33b) into (34a) to yield

$$E_{\mathcal{L}} + Z_{\mathcal{L}}I(\mathcal{L}) = \Phi_{11}V(0) + \Phi_{12}I(0). \tag{36}$$

Substituting from (34b)

$$V(0) = \Phi_{21}^{-1}I(\mathcal{L}) - \Phi_{21}^{-1}\Phi_{22}I(0) \tag{37}$$

into (36), rearranging and multiplying on the left by Φ_{21} yields

$$(-\Phi_{21}Z_{\mathcal{L}} + \Phi_{21}\Phi_{11}\Phi_{21}^{-1})I(\mathcal{L}) + (\Phi_{21}\Phi_{12} - \Phi_{21}\Phi_{11}\Phi_{21}^{-1}\Phi_{22})I(0) = \Phi_{21}E_{\mathcal{L}}. \tag{38}$$

Using Identity 2 and Identity 4 in (38) yields

$$(\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})I(\mathcal{L}) + I(0) = -\Phi_{21}E_{\mathcal{L}}. \tag{39}$$

Equations (35) and (39) can be arranged in matrix form as

$$\begin{bmatrix} (\Phi_{21}Z_0 - \Phi_{22}) & 1_n \\ 1_n & (\Phi_{21}Z_{\mathcal{L}} - \Phi_{22}) \end{bmatrix} \begin{bmatrix} I(0) \\ I(\mathcal{L}) \end{bmatrix} = \begin{bmatrix} \Phi_{21} & E_0 \\ -\Phi_{21} & E_{\mathcal{L}} \end{bmatrix} \tag{40}$$

which has a highly sparse coefficient matrix with $2(n^2 - n)$ of the total $4n^2$ elements identically zero. Equation (40) can also be solved explicitly for $I(0)$ and $I(\mathcal{L})$ as

$$\{1_n - (\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})(\Phi_{21}Z_0 - \Phi_{22})\}I(0) = -\Phi_{21}E_{\mathcal{L}} - (\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})\Phi_{21}E_0 \tag{41a}$$

$$I(\mathcal{L}) = -(\Phi_{21}Z_0 - \Phi_{22})I(0) + \Phi_{21}E_0. \tag{41b}$$

$V(x)$ and $I(x)$ at any point along the line can be found from (3) once $I(0)$ is obtained from the solution of (40) or (41a) and $V(0)$ is obtained from (33a).

The identities have reduced the number of redundant matrix multiplications and, moreover, only two of the four matrix chain parameter submatrices are required to be computed; Φ_{21} and Φ_{22} [see (15)]. Reducing the number of required matrix multiplications is an important consideration in numerical machine computation. For example, n^3 operations (multiplications or divisions) are required to multiply two "full" $n \times n$ matrices which is the minimum number of operations required to invert a "full" $n \times n$ matrix [6]. The solution of a set of n equations in n unknowns by Gauss elimination requires $n^3/3 + n^2 - n/3$ operations or $n^3/3$ for large n [6]. Forming $\Phi_{21}Z_0$ and $\Phi_{21}Z_{\mathcal{L}}$ require $2n^3$ operations. Therefore, solution of (40) requires on the order of $(2n)^3/3 + 2n^3 = 14n^3/3$ total operations (neglecting the n^2 operations required to form $\Phi_{21}E_0$ and $\Phi_{21}E_{\mathcal{L}}$). Solution of (41a) requires an additional n^3 operations for the multiplication of $(\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})(\Phi_{21}Z_0 - \Phi_{22})$ and n^3 operations for the multiplication of $(\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})\Phi_{21}$. Thus the total number of operations required to solve (41) is on the order of $n^3/3 + 4n^3 = 13n^3/3$ operations. Therefore, it may be more efficient to solve (40) rather than (41) since almost the same number of operations are involved and the sparsity of (40) can be used to reduce the required number of operations even further.

V. SUMMARY

Certain matrix identities among the submatrices of the chain parameter matrix for multiconductor transmission lines have been shown. The identities reduce to familiar results for the two-conductor transmission line where the submatrices become complex scalars. The order of multiplication of the submatrices must be carefully adhered to for the multiconductor case since the various matrix products do not in general commute. A set of matrix equations which incorporate the termination networks for the total solution of the line currents was formulated. Using the matrix identities the coefficient matrix was reduced to a highly sparse and efficient form.

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Green's Function in a Region with Inhomogeneous, Isotropic Dielectric Media

MASANORI KOBAYASHI

***Abstract*—The reciprocity relation satisfied by the Green's function for the inhomogeneous partial differential equation in a multidielectric region with inhomogeneous, isotropic media is derived by using Green's theorem.**

I. INTRODUCTION

The calculation of the parameters of a microstrip line based on a TEM approximation is useful for the design of microwave integrated circuit structures. The parameters often can be calculated by variational techniques using Green's functions [1]–[5]; however, the Green's function satisfying the boundary conditions must be obtained first.

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The author is with the Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki, 316 Japan.

Recently, the validity of the reciprocity relation satisfied by the Green's function for Poisson's equation in a two-dielectric region containing homogeneous, isotropic media was pointed out [6]. On the other hand, the reciprocity relation for the Green's function in a multidielectric region containing inhomogeneous, isotropic media has not been discussed in the literature.

The purpose of this paper is to show that the reciprocity relation for the Green's function in such a region holds for the Dirichlet, Neumann, and mixed problems by using Green's theorem.

II. GREEN'S THEOREM

In the three-dimensional space, consider the region R containing the dielectric medium whose permittivity $\epsilon(r)$ is a function of position r and continuous with continuous first-order partial derivatives. Let $u(r)$ and $w(r)$ be two scalar functions of position in the region R. Applying the divergence theorem to the vector function $\epsilon(r)u(r)\cdot\nabla w(r)$, and using the method shown by Collin [5, p. 556], we obtain Green's theorem

$$\iiint_R [u(r)\nabla\cdot\{\epsilon(r)\nabla w(r)\} - w(r)\nabla\cdot\{\epsilon(r)\nabla u(r)\}]\,dv = \iint_S \epsilon(r)\{u(r)\nabla w(r) - w(r)\nabla u(r)\}\cdot n\,da \quad (1)$$

where n denotes the unit vector of the external normal to the surface S enclosing the region R.

III. RECIPROCITY RELATION FOR GREEN'S FUNCTIONS

Consider a p-dielectric region R in the three-dimensional space. The region R enclosed by the surface S is composed of the regions $R_i(i = 1,2,\cdots,p)$. Let the region R_i be filled with the inhomogeneous, isotropic dielectric medium of permittivity $\epsilon(r)$, which is a function with continuous first-order partial derivatives, and be bounded by the surface S_i which is piecewise smooth. Using the abbreviated notation, the source point (x_0,y_0,z_0) will be designated by r_0; the observation point (x,y,z), by r; and the three-dimensional delta function, by $\delta(r - r_0)$.

The Green's function $G(r;r_0)$ which is used in calculating the parameters of a microstrip line based on a TEM approximation is defined as a solution of the three-dimensional inhomogeneous partial differential equation modified the (3) in [2]

$$\nabla\cdot\{\epsilon(r)\nabla G\} = -\delta(r - r_0) \quad (2)$$

subject to the homogeneous boundary condition

$$\alpha(r)(\nabla G)\cdot n + \beta(r)G = 0 \quad \text{on} \quad S, S = \bigcup_{i=1}^{p} S_i - \bigcap_{i=1}^{p} S_i \quad (3)$$

and the boundary conditions at the interfaces $S_i \cap S_j(i,j = 1,2,\cdots,p$, where $i \neq j)$

$$G\,|_{S_i\cap S_j, r\epsilon S_i} = G\,|_{S_j\cap S_i, r\epsilon S_j} \quad (4)$$

$$\epsilon_i(r)(\nabla G)\cdot n_i\,|_{S_i\cap S_j, r\epsilon S_i} = -\epsilon_j(r)(\nabla G)\cdot n_j\,|_{S_j\cap S_i, r\epsilon S_j} \quad (5)$$

where

r_0 $= r_i, G = G_i \equiv G(r;r_i)$ when $r_0 \in R_i(i = 1,2,\cdots,p)$;
$\epsilon(r)$ $= \epsilon_i(r)$ when $r \in R_i(i = 1,2,\cdots,p)$;
n the unit vector of the external normal to S;
n_i the unit vector of the external normal to $S_i(i = 1,2,\cdots,p)$;

the Dirichlet problem:

$$\alpha(r) = 0, \beta(r) \neq 0 \text{ on } S \quad (6)$$

the Neumann problem:

$$\alpha(r) \neq 0, \beta(r) = 0 \text{ on } S \quad (7)$$

the mixed problem:

$$\alpha(r) \neq 0, \beta(r) \neq 0 \text{ on } S. \quad (8)$$

Replacing $u(r)$ by $G_i \equiv G(r;r_i)$, and $w(r)$ by $G_k \equiv G(r;r_k)$ in Green's theorem (1), we obtain the following equation for the region $R_m(m = 1,2,\cdots,p)$:

$$\iiint_{R_m} [G_i\nabla\cdot\{\epsilon_m(r)\nabla G_k\} - G_k\nabla\cdot\{\epsilon_m(r)\nabla G_i\}]\,dv = \iint_{S_m} \epsilon_m(r)\{G_i\nabla G_k - G_k\nabla G_i\}\cdot n_m\,da. \quad (9)$$

Substituting (2) into the left-hand side of (9) for the region R_i yields $G(r_i;r_k)$. Let us denote the surface $S_i - \Sigma_i(t \neq i)$, as S^i, and denote $S^i = {}_\alpha S^i \cup {}_\beta S^i$, where

$$\Sigma_i\,(t \neq i) \equiv \bigcup_{t=1;(t\neq i)}^{p} (S_i \cap S_t).$$

the surfaces ${}_\alpha S^i$ and ${}_\beta S^i$ denote the subsurface of S^i on which $\alpha(r) \neq 0$ and that on which $\beta(r) \neq 0$, respectively. The integrand in the right-hand side of (9) for the region R_i becomes zero on the surfaces ${}_\alpha S^i$ and ${}_\beta S^i$ from (4) and (5). Therefore, since the surface integral on S^i becomes zero, (9) for the region R_i becomes

$$G(r_i;r_k) = \iint_{\Sigma_i(t\neq i)} \epsilon_i(r)(G_i\nabla G_k - G_k\nabla G_i)\cdot n_i\,da. \quad (10)$$

In a similar procedure, we obtain the following equations for the regions R_k and $R_j(j = 1,2,\cdots,p$, where $j \neq i,k)$:

$$G(r_k;r_i) = -\iint_{\Sigma_k(t\neq k)} \epsilon_k(r)(G_i\nabla G_k - G_k\nabla G_i)\cdot n_k\,da \quad (11)$$

$$0 = \iint_{\Sigma_j(t\neq j)} \epsilon_j(r)(G_i\nabla G_k - G_k\nabla G_i)\cdot n_j\,da \quad (j = 1,2,\cdots,p, j \neq i,k). \quad (12)$$

Summing (12) over all j, and rearranging those surface integrals by using (4) and (5), we obtain

$$\iint_{\Sigma_i(t\neq i,k)} \epsilon_i(r)(G_i\nabla G_k - G_k\nabla G_i)\cdot n_i\,da = -\iint_{\Sigma_k(t\neq i,k)} \epsilon_k(r)(G_i\nabla G_k - G_k\nabla G_i)\cdot n_k\,da. \quad (13)$$

Adding the surface integral on $S_i \cap S_k$ to that in the left-hand side of (13), and adding the surface integral on $S_k \cap S_i$ to that in the right-hand side of (13) by using (4) and (5), we obtain the reciprocity relation

$$G(r_i;r_k) = G(r_k;r_i). \quad (14)$$

Therefore, we find that the reciprocity relation (14) satisfied by the Green's function for the inhomogeneous partial differential equation (2) is valid for the boundary value problems (6)–(8).

IV. CONCLUSION

It has been presented that the reciprocity relation for the Green's function is valid for the Dirichlet, Neumann, and mixed problems. The reciprocity relation obtained here will be useful in determining the Green's function for a microstrip transmission line with multidielectric layers whose permittivities are functions of position. Further work is in hand to calculate the parameters of such a microstrip transmission line by the variational technique using Green's function.

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On the Question of Computation of the Dyadic Green's Function at the Source Region in Waveguides and Cavities

Y. RAHMAT-SAMII

Abstract—**The derivation of the dyadic Green's function for rectangular waveguides and cavities is approached systematically by using the theory of distributions. It is shown that, in order to obtain a complete solution for the field distribution in the entire structure, one must add an additional term to the classical expansion which is valid only outside the source region.**

I. INTRODUCTION

In a recent paper, Collin [1] has discussed the question of incompleteness of the E and H modes in the source region of a waveguide, and has shown that an additional term must be added to the classical representation of the field expression in order to derive a complete solution that is valid both in the source and source-free region of the waveguide. Tai [2] has also noted the difficulties involved in the computation of the dyadic Green's function, and has presented a solution based upon the use of eigenvector functions M and N. Neither of the preceding methods shares the simplicity of the conventional techniques used in solving the source-free waveguide problems, nor gives a clear picture of removing the difficulties involved in computing the dyadic Green's function in the source region.

The purpose of this short paper is to develop a systematic and novel approach for determining the dyadic Green's function and, consequently, the field distribution in the entire region of rectangular waveguides and cavities. It is shown that if one carefully defines the derivatives in the distribution sense and applies the correct completeness property of the modes, it is then possible to construct the complete solution for the entire structure just by employing the scalar eigenfunctions of the Helmholtz equation.

II. COMPUTATION OF THE DYADIC GREEN'S FUNCTION $\mathbf{G}_e$ AND FIELD DISTRIBUTION IN WAVEGUIDES

It is well known that the electric field excited by an electric current source can be expressed in terms of the dyadic Green's function $\mathbf{G}_e(r \mid r')$ as

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The author is with the Department of Electrical Engineering, University of Illinois at Urbana-Champaign, Urbana, Ill. 61801.

$$E(r) = -j\omega\mu \int \mathbf{G}_e(r \mid r') \cdot J(r')\, dv'. \tag{1}$$

One may notice from the preceding equation that care must be exercised in deriving $\mathbf{G}_e(r \mid r')$ at the source region if one wants to perform the preceding integration correctly. It is our purpose to derive the correct $\mathbf{G}_e$ in this section. The geometry of the problem with dimensions a and b along x and y axes, respectively, is a perfectly conducting rectangular waveguide aligned along the z axis and excited by a finite-volume electric current source.

To attack the problem, we write the Maxwell's equation and necessary boundary conditions as

$$\left.\begin{aligned} \nabla \times H &= J + j\omega\epsilon E \\ \nabla \times E &= -j\omega\mu H \end{aligned}\right\}, \quad \text{in the waveguide} \tag{2a}$$

$$\hat{n} \times E = 0, \qquad \text{on the wall.} \tag{2b}$$

It is assumed that the waveguide is filled with a homogeneous and isotropic material. In order to obtain the unique solution, the field components must also satisfy the Sommerfeld radiation condition along the z axis. From (2) one easily concludes that

$$\nabla \cdot H = 0, \qquad \text{in the waveguide} \tag{3a}$$

$$\hat{n} \cdot H = 0, \qquad \text{on the wall.} \tag{3b}$$

It is also trivial to show that E and H fields can be separated as follows:

$$\nabla \times \nabla \times E - k^2 E = -j\omega\mu J \tag{4a}$$

$$\nabla \times \nabla \times H - k^2 H = \nabla \times J. \tag{4b}$$

In order to solve the preceding boundary-value problem, we introduce the electric-type dyadic Green's function $\mathbf{G}_e(r \mid r')$ and magnetic-type dyadic Green's function $\mathbf{G}_m(r \mid r')$ by the following definitions:

$$\nabla \times \nabla \times \mathbf{G}_e - k^2 \mathbf{G}_e = \mathbf{I}\delta(r - r') \tag{5a}$$

$$\hat{n} \times \mathbf{G}_e = 0, \qquad \text{on the wall} \tag{5b}$$

and

$$\nabla \times \nabla \times \mathbf{G}_m - k^2 \mathbf{G}_m = \nabla \times \mathbf{I}\delta(r - r') \tag{6a}$$

$$\left.\begin{aligned} \hat{n} \cdot \mathbf{G}_m &= 0 \\ \hat{n} \times \nabla \times \mathbf{G}_m &= 0 \end{aligned}\right\}, \qquad \text{on the wall} \tag{6b}$$

where $\mathbf{I}$ is a unit dyadic. Notice that $\mathbf{G}_e$ and $\mathbf{G}_m$ are also subjected to the Sommerfeld radiation condition as discussed by Tai [3].

The first equation of (2a) shows that the following relation holds between $\mathbf{G}_m$ and $\mathbf{G}_e$:

$$k^2 \mathbf{G}_e = \nabla \times \mathbf{G}_m - \mathbf{I}\delta(r - r'). \tag{7}$$

Note that (1) can be derived by applying the Green's theorem between E and $\mathbf{G}_e$, and by making use of the boundary and radiation conditions.

The central issue regarding the completeness of modal representation of the dyadic Green's function is that $\nabla \cdot \mathbf{G}_e \neq 0$ in the waveguide. That $\nabla \cdot \mathbf{G}_e \neq 0$ is evident if the divergence operator is applied to both sides of (5a) and by recalling that $\nabla \cdot \mathbf{I}\delta(r - r') \neq 0$. Since $\nabla \cdot \mathbf{G}_e$ is not zero in the entire waveguide, $\mathbf{G}_e$ cannot be constructed in terms of the superposition of E modes alone (see, for instance, Goubau [4]). An additional term is necessary to obtain the complete form of $\mathbf{G}_e$. In what follows, $\mathbf{G}_e$ will be found by using (7) after the complete form of $\mathbf{G}_m$ is derived. It should be noted that in contrast to $\mathbf{G}_e$, $\mathbf{G}_m$ can be expressed in terms of H modes only, since $\nabla \cdot \mathbf{G}_m = 0$ everywhere.

In the following, a procedure is presented for constructing $\mathbf{G}_m$ which is believed to be simpler than Tai's method. Tai [2] used eigenvector functions M and N, which are divergenceless vectors to expand $\nabla \times \mathbf{I}\delta(r - r')$, and then he constructed $\mathbf{G}_m$ from (6a). In our procedure, we use the fact that $\nabla \cdot \mathbf{G}_m = 0$ and $\nabla \times \nabla \times =$